Contents

1 Homotopy Coherence

No, I have not discovered the model structure for quasi-categories in the 1980's. I became interested in quasi-categories (without the name) around 1980 after attending a talk by Jon Beck on the work of Boardman and Vogt. I wondered if category theory could be extended to quasi-categories. In my mind, a crucial test was to show that a quasi-category is a Kan complex if its homotopy category is a groupoid. All my attempts at showing this have failed for about 15 years, until I stopped trying hard! I found a proof after extending to quasi-categories a few basic notions of category theory. This was around 1995. The model structure for quasi-categories was discovered soon after. I did not publish it immediately because I wanted to show that it could be used for proving something new in homotopy theory. I am a bit of a perfectionist (and overly ambitious?). I was hoping to develop a synthesis between category theory and homotopy theory (hence the name quasi-categories). I met Lurie at a conference organised by Carlos Simpson in Nice (in 2001?). I gave a talk on the model structure and Lurie asked for a copy of my notes afterward. I intuitively understood that he could develop the theory of quasi-categories more and better than I could. He was young and a better mathematician than I was. I do not regret it.

- André Joyal on the history of the Joyal model structure, see Math Overflow Answer

1.1 Review

Last time we constructed an adjunction

$$\begin{tikzcd} \mathrm{Cat}_\Delta \arrow[r, phantom, "\perp"] & \mathrm{Set}_\Delta \arrow[l, bend right, "\mathfrak{C}"'] \arrow[l, leftarrow, bend left, "\mathfrak{N}"] \end{tikzcd}$$

where we called $\mathfrak{N}$ the homotopy coherent nerve functor, while $\mathfrak{C}$ was referred to as the rigidification functor. The construction of the above adjoint pair was done in two steps:

- First we constructed a functor $S\colon \Delta \to \mathrm{Set}_\Delta$.
- We left Kan extended S along the Yoneda embedding $\mathcal{Y}_\Delta$ to obtain the unique cocontinuous extension $\mathfrak{C}$ of S:

$$\begin{tikzcd} \Delta \arrow[r, "S"] \arrow[d, hook, "\mathcal{Y}_\Delta"'] & \mathrm{Cat}_\Delta \\ \mathrm{Set}_\Delta \arrow[ur, dotted, "\mathfrak{C}"'] & \end{tikzcd}$$

By the nerve realization Theorem, we know that $\mathfrak{C} \dashv \mathfrak{N}$ where

$$\mathfrak{N}\colon \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta, \qquad \mathscr{F} \mapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(S[-], \mathscr{F})$$

Let us recall this construction: We start by building the functor $S\colon \Delta \to \mathrm{Cat}_\Delta$ valued in the category of simplicially enriched categories. For any finite ordinal $[n]$ the functor S should spit out a simplicially enriched category $S[n]$ which has

- its objects given by the set $[n] = \{0, \dots, n\}$.
- its morphism simplicial sets, for objects $i, j \in [n]$, are given by

$$S[n](i,j) := \mathcal{N}(P_{i\to j})$$

  where $\mathcal{N}\colon \mathrm{Cat} \to \mathrm{Set}_\Delta$ is the nerve functor for (ordinary) categories, while $P_{i\to j}$ is the category whose objects are given by subsets

$$\{i,j\} \subset T \subset [i,j]$$

  In particular, $P_{i\to j} = \emptyset$ if $i > j$. For a second such subset T' as above, there is an arrow $T \to T'$ if and only if $T \subset T'$.
- The composition morphisms

$$S[n](i,j) \times S[n](j,k) \longrightarrow S[n](i,k)$$

  for $i, j, k \in [n]$ are induced by taking unions.

- The identity arrow picked out by

$$\Delta^0 \to S[n](i,i)$$

  is given by the singleton $\{i\}$.

This neatly assembles into a simplicially enriched category $S[n]$. Moreover, the coface and codegeneracy maps induced by the functor (that we want to construct here) $S\colon \Delta \to \mathrm{Cat}_\Delta$

$$S[n-1] \xrightarrow{\quad d^i \quad} S[n]$$

$$S[n+1] \xrightarrow{\quad s^i \quad} S[n]$$

do the "obvious" things: On objects these simplicially enriched functors act via the functions of sets $d^i\colon [n-1] \to [n]$ and $s^i\colon [n+1] \to [n]$. For $k, l \in [n]$ their actions on the morphism simplicial sets

$$S[n-1](k,l) \xrightarrow{\quad d^i \quad} S[n](d^i(k), d^i(l))$$

$$S[n+1](k,l) \xrightarrow{\quad s^i \quad} S[n](s^i(k), s^i(l))$$

is induced by applying the maps d^i and s^i to subsets $\{k,l\} \subset T \subset [k,l] \subset [n-1]$ and $\{k,l\} \subset U \subset [k,l] \subset [n+1]$ to obtain

$$\{d^i(k), d^i(l)\} \subset d^i(T) \subset [d^i(k), d^i(l)] \subset [n]$$
$$\{s^i(k), s^i(l)\} \subset s^i(U) \subset [s^i(k), s^i(l)] \subset [n]$$

Thus it is that we have constructed a functor $S\colon \Delta \to \mathrm{Cat}_\Delta$. The category $P_{0\to 4}$ may be depicted as follows:

$$\begin{tikzcd}
& \{0,4\} \arrow[rr] \arrow[dl] \arrow[dd] \arrow[dr, dotted] \arrow[ddrr, dotted] \arrow[dddl, dotted] \arrow[dddr, dotted] & & \{0,1,4\} \arrow[dl] \arrow[dd] \arrow[dddl, dotted] \\
\{0,3,4\} \arrow[rr] \arrow[dd] & & \{0,1,3,4\} \arrow[dd] & \\
& \{0,2,4\} \arrow[rr] \arrow[dl] \arrow[dr, dotted] & & \{0,1,2,4\} \arrow[dl] \\
\{0,2,3,4\} \arrow[rr] & & \{0,1,2,3,4\} &
\end{tikzcd}$$

So we see that $\mathcal{N}(P_{0\to 4})$ is isomorphic to $\Delta^1 \times \Delta^1 \times \Delta^1$ (the simplicial cube). More generally, $\mathcal{N}(P_{i\to j}) \cong (\Delta^1)^{\times(j-i-1)}$. This plus the ominous nerve realization paradigm (see Nerve Realization) we immediately obtain: We left Kan extend $S\colon \Delta \to \mathrm{Cat}_\Delta$ along the Yoneda embedding:

$$\begin{tikzcd}
\Delta \arrow[r, "S"] \arrow[d, hook, "\mathcal{Y}_\Delta"'] & \mathrm{Cat}_\Delta \\
\mathrm{Set}_\Delta \arrow[ur, dotted, "\mathfrak{C}"'] &
\end{tikzcd}$$

Recall here that $\mathfrak{C} := \mathrm{Lan}_{\mathcal{Y}_\Delta}(S)$ thus defined is the unique cocontinuous extension of S i.e. we have $\mathfrak{C}|_\Delta \cong S$ or equivalently

$$\mathfrak{C}(\Delta^n) \cong S[n]$$

functorially in $[n] \in \Delta$. In particular, for an arbitrary simplicial set X, we know by the density theorem (see Density) that there is a category $\mathscr{I}_X$ along with a functor $\xi\colon \mathscr{I}_X \to \Delta$ such that

$$X \cong \operatorname*{colim}_{i \in \mathscr{I}_X} \Delta^{\xi(i)}$$

The image of X under $\mathfrak{C}$ is then given by

$$\mathfrak{C}(X) \cong \operatorname*{colim}_{i \in \mathscr{I}_X} S(\xi(i))$$

The nerve realization paradigm then says this construction is left adjoint to the homotopy coherent nerve functor

$$\mathfrak{N}\colon \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta, \qquad \mathscr{F} \mapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(S[-], \mathscr{F})$$

A homotopy coherent diagram of shape $\mathscr{I} \in \mathrm{Set}_\Delta$ in some simplicially enriched category $\mathscr{F}$ is a map of simplicial sets

$$\mathscr{I} \to \mathfrak{N}(\mathscr{F})$$

Equivalently, by exploiting the adjunction $(\mathfrak{C} \dashv \mathfrak{N})$ we could also define the same homotopy coherent diagram via a simplicially enriched functor

$$\mathfrak{C}(\mathscr{I}) \to \mathscr{F}$$

Let $\mathbb{N}$ be the category of natural numbers and view it as a simplicial set $\Delta^{\mathrm{op}} \to \mathrm{Set}$ by taking its (ordinary) nerve (the nerve of $\mathbb{N}$ will again be denoted by $\mathbb{N}$). A map of simplicial sets

$$\mathbb{N} \to \mathfrak{N}(\mathrm{Spaces})$$

is then, by the above definition, a homotopy coherent diagram of shape $\mathbb{N}$ in Spaces. Now does this agree with the very very informal definition of a homotopy coherent diagram of shape $\mathbb{N}$ that we give in the motivational part of the last lecture? Yes, it does.

Proof. A morphism

$$\mathbb{N} \to \mathfrak{N}\mathrm{Spaces}$$

is an assignment

- of 0-simplices

  $$\mathbb{N}_0 \ni i \mapsto X_i \in \mathfrak{N}\mathrm{Spaces}_0 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}(\Delta^0), \mathrm{Spaces})$$

  which simply boils down to X_i being a space i.e. an object in Spaces. This can be mysteriously depicted by:

  $$\underset{X_0}{\{0\}}$$

  where 0 denotes i to make it less confusing later on.

- of 1-simplices

  $$\mathbb{N}_1 \ni (i \to j) \longmapsto f_{i,j} \in \mathfrak{N}\mathrm{Spaces}_1 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}\Delta^1, \mathrm{Spaces})$$

  This means the arrow $(i \to j)$ is mapped to the simplicially enriched functor $f_{i,j} \colon \mathfrak{C}(\Delta^1) \to \mathrm{Spaces}$. But such a diagram is determinded already by the datum on morphism simplicial sets

  $$\mathfrak{C}\Delta^1(0,1) \cong \Delta^0 \xrightarrow{f_{i,j}} \mathrm{Spaces}(X_i, X_j)$$

  In other words, $f_{i,j}$ is a map of spaces $X_i \to X_j$. This can also be depicted (mysteriously so, but it will make sense soon enough) by:

  $$\underset{f_{0,1}}{\{0,1\}}$$

  where $i = 0, j = 1$.

- of 2-simplices

  $$\mathbb{N}_2 \ni (i \to j,\ j \to k) \longmapsto h_{i,j,k} \in \mathfrak{N}\mathrm{Spaces}_2 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}\Delta^2, \mathrm{Spaces})$$

  The action of morphism simplicial sets of this simplicially enriched functor $h_{i,j,k}$ is given by

  $$\mathfrak{C}\Delta^2(0,2) \cong \Delta^1 \xrightarrow{h_{i,j,k}} \mathrm{Spaces}(X_i, X_k)$$

  $$\mathfrak{C}\Delta^2(0,1) \cong \Delta^0 \xrightarrow{f_{i,j}} \mathrm{Spaces}(X_i, X_j)$$

  $$\mathfrak{C}\Delta^2(1,2) \cong \Delta^0 \xrightarrow{f_{j,k}} \mathrm{Spaces}(X_j, X_k)$$

By using commutativity of

$$\begin{tikzcd}
\mathfrak{C}\Delta(0,1) \times \mathfrak{C}\Delta^2(1,2) \arrow[r] \arrow[d, hook, "\cup"'] & \mathrm{Spaces}(X_i, X_j) \times \mathrm{Spaces}(X_j, X_k) \arrow[d] \\
\mathfrak{C}\Delta^2(0,2) \arrow[r] & \mathrm{Spaces}(X_i, X_k)
\end{tikzcd}$$

we obtain that $h_{i,j,k} \colon \Delta^1 \to \mathrm{Spaces}(X_i, X_k)$ is equivalently a homotopy

$$\begin{tikzcd}
& X_j \arrow[dr, "f_{j,k}"] & \\
X_i \arrow[ur, "f_{i,j}"] \arrow[rr, "f_{i,k}"'] \arrow[rr, phantom, shift left=4, "h_{i,j,k}"] & & X_k
\end{tikzcd}$$

Mysteriously (or not so much anymore) again:

$$\underset{f_{0,2}}{\{0,2\}} \xrightarrow{h_{0,1,2}} \underset{f_{1,2}f_{0,1}}{\{0,1,2\}}$$

- of 3-simplices

$$\mathbb{N}_3 \ni (i \to j,\ j \to k,\ k \to l) \longmapsto H_{i,j,k,l} \in \mathfrak{N}\mathrm{Spaces}_3 = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}\Delta^3, \mathrm{Spaces})$$

And in an analogous manner as before, the 3-simplex $H_{i,j,k,l}$ will correspond to a 2-homotopy (homotopy between homotopies) as depicted (where $i = 0, j = 1, k = 2, l = 3$):

$$\begin{tikzcd}
\underset{f_{0,3}}{\{0,3\}} \arrow[r, "h_{0,1,3}"] \arrow[d, "h_{0,2,3}"'] \arrow[dr, phantom, "H_{0,1,2,3}"] & \underset{f_{1,3}f_{0,1}}{\{0,1,3\}} \arrow[d, "h_{1,2,3}f_{0,1}"] \\
\underset{f_{2,3}f_{0,2}}{\{0,2,3\}} \arrow[r, "f_{2,3}h_{0,1,2}"'] & \underset{f_{2,3}f_{1,2}f_{0,1}}{\{0,1,2,3\}}
\end{tikzcd}$$

- of 4-simplices ... Here we can draw the following picture, which should make perfect sense at this point:

$$\begin{tikzcd}
& \underset{f_{1,4}f_{0,1}}{\{0,1,4\}} \arrow[rr, "h_{1,2,4}f_{0,1}"] \arrow[dd] & & \underset{f_{2,4}f_{1,2}f_{0,1}}{\{0,1,2,4\}} \arrow[dd, "h_{2,3,4}f_{1,2}f_{0,1}"] \\
\underset{f_{0,4}}{\{0,4\}} \arrow[ur, "h_{0,1,4}"] \arrow[rr, "h_{0,2,4}"] \arrow[dd] & & \underset{f_{2,4}f_{0,2}}{\{0,2,4\}} \arrow[ur, "f_{2,4}h_{0,1,2}"] \arrow[dd] & \\
& \underset{f_{3,4}f_{1,3}f_{0,1}}{\{0,1,3,4\}} \arrow[rr, "f_{3,4}h_{1,2,3}f_{0,1}"] & & \underset{f_{3,4}f_{2,3}f_{1,2}f_{0,1}}{\{0,1,2,3,4\}} \\
\underset{f_{3,4}f_{0,3}}{\{0,3,4\}} \arrow[ur, "f_{3,4}h_{1,3,4}"] \arrow[rr, "f_{3,4}h_{0,2,3}"] & & \underset{f_{3,4}f_{2,3}f_{0,2}}{\{0,2,3,4\}} \arrow[ur, "f_{3,4}f_{2,3}h_{0,1,2}"] &
\end{tikzcd}$$

where every face is filled by the respective $H_{i,j,k,l}$. Moreover, this cube has a volume or filling given by some $F_{0,1,2,3,4}\colon \mathfrak{C}\Delta^4 \to \text{Spaces}$.

- and so on...

$\square$

1.2 Dwyer-Kan-Bergner and Joyal model structures

Note that a simplicially enriched category $\mathscr{F}$ has an induced homotopy category $\text{Ho}\mathscr{F}$ which has the same objects as $\mathscr{F}$, yet its set of morphisms from X to Y is given by the path components

$$\pi_0\mathscr{F}(X,Y)$$

where $\pi_0\colon \text{Set}_\Delta \to \text{Set}$ is the functor given by

$$X_\bullet \longmapsto \text{coeq}\Big(X_1 \rightrightarrows X_0\Big)$$

In particular, composition an units on $\text{Ho}\mathscr{F}$ are induced by $\mathscr{F}$ by simply applying π_0 to the corresponding datum. In particular, for a simplicially enriched functor $F\colon \mathscr{F} \to \mathscr{F}'$ the above construction induces a functor of ordinary categories $\pi_0 F\colon \text{Ho}\mathscr{F} \to \text{Ho}\mathscr{F}'$. Let $\mathscr{F}$ and $\mathscr{F}'$ be simplicially enriched categories. A functor $F : \mathscr{F} \to \mathscr{F}'$ is called a Dwyer-Kan equivalence if it satisfies the following two conditions:

- it is (homotopically) essentially surjective i.e. $\pi_0 F\colon \text{Ho}\mathscr{F} \to \text{Ho}\mathscr{F}'$ is essentially surjective.

- it is ∞-full and faithful i.e. for any pair of objects $X, Y \in \mathscr{F}$ the action of F on Hom-simplicial sets

$$\mathscr{F}(X,Y) \xrightarrow{\sim} \mathscr{F}'(FX,FY)$$

is a weak homotopy equivalence of simplicial sets.

There exists a model structure for Cat_Δ, called Dwyer-Kan-Bergner model structure, which we will denote by $(\mathrm{Cat}_\Delta)_{\mathrm{DKB}}$:

- weak equivalences in this model category arer given by Dwyer Kan equivalences.

- for fibrations and cofibrations check out the Nlab DKB model structure.

Fibrant objects in $(\mathrm{Cat}_\Delta)_{\mathrm{DKB}}$ are precisely Kan complex enriched categories, i.e. categories enriched over ∞-groupoids. There exists a model structure on Set_Δ called Joyal model structure, which we will denote by $(\mathrm{Set}_\Delta)_{\mathrm{Joyal}}$:

- cofibrations are monomorphisms.

- weak equivalences are those maps of simplicial sets $X \to Y$ which are mapped to Dwyer Kan equivalences by the rigidification functor $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$.

Fibrant objects in $(\mathrm{Set})_{\mathrm{Joyal}}$ are precisely quasicategories. The important result of this section is that both $(\mathrm{Cat}_\Delta)_{\mathrm{DKB}}$ and $(\mathrm{Set}_\Delta)_{\mathrm{Joyal}}$ are models for ∞-categories, which is witnessed by: There is a Quillen equivalence

$$\begin{tikzcd} (\mathrm{Cat}_\Delta)_{\mathrm{DKB}} \arrow[r, shift right, "\mathfrak{N}"'] \arrow[r, phantom, "\simeq_{\text{Quillen}}"] & (\mathrm{Set}_\Delta)_{\mathrm{Joyal}} \arrow[l, shift right, "\mathfrak{C}"'] \end{tikzcd}$$

This means that, up to equivalence, any Kan complex enriched category is the rigidification of a quasi-category and any quasi-category is the homotopy coherent nerve of a Kan complex enriched category. Let us utilize the above equivalence:

- $\infty\mathrm{Grpd} \coloneqq \mathfrak{N}(\mathrm{Kan})$ is defined to be the ∞-category of ∞-groupoids.

- Consider the Kan complex enriched category $\mathrm{QCat}^{\mathrm{core}}$ which has objects quasicategories and the corresponding morphism simplicial sets are given by

$$\mathrm{core}(\mathrm{Fun}(X,Y)) \in \mathrm{Kan}$$

for X, Y quasicategories. Here core is a right adjoint functor fitting into the adjunction:

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, shift right, "\mathrm{core}"'] \arrow[r, phantom, "\perp"] & \mathrm{Kan} \arrow[l, shift right, "\mathrm{forget}"'] \end{tikzcd}$$

More concretely, given a simplicial set X, the core of X, denoted as $\mathrm{core}(X)$, is defined as the largest sub-simplicial set $Y \subseteq X$ such that Y is a Kan complex. Having this, we define

$$\infty\mathrm{Cat} \coloneqq \mathfrak{N}(\mathrm{QCat}^{\mathrm{core}})$$

Strictly speaking ∞Cat should really be an $(\infty, 2)$-category instead of just an $(\infty, 1)$-category as defined above; we do not have the tools however to talk about this, so we will avoid this conversation altogether.

- Let R be a ring and $\mathrm{Ch}(R)$ be the category of chain complexes over R. This category is Kan complex enriched as we will see in a moment. First a little bit of preparation is needed: There is a functor

$$C_\bullet^{\mathrm{simp}} \colon \Delta \to \mathrm{Ch}(\mathrm{Ab}), \qquad [n] \mapsto C_\bullet^{\mathrm{simp}}(\Delta^n)$$

where $C_\bullet^{\mathrm{simp}}(\Delta^n)$ is the chain complex of abelian groups which in degree m is given by

$$\mathbb{Z}\big[\{\text{non-degenerate m-simplices of } \Delta^n\}\big]$$

Here $\mathbb{Z}[-]$ is the free abelian group functor:

$$\begin{tikzcd} \mathrm{Ab} \arrow[r, shift left, "\text{forget}"] \arrow[r, phantom, "\top"] & \mathrm{Set} \arrow[l, shift left, "{\mathbb{Z}[-]}"] \end{tikzcd}$$

and the differentials are given by

$$\partial_m \coloneqq \sum_{i=0}^{m} (-1)^i d_i$$

with $d_i \colon \Delta^n_m \to \Delta^n_{m-1}$ the induced face maps. With this we can verify that $\mathrm{Ch}(R)$ is Kan complex enriched by means of the construction:

$$\mathrm{Ch}(R)^{\mathrm{simp}}(C_\bullet, D_\bullet) \coloneqq \mathrm{Hom}_{\mathrm{Ch}(R)}(C_\bullet \otimes_{\mathbb{Z}} C_\bullet^{\mathrm{simp}}(\Delta^n), D_\bullet)$$

It can be shown that $\mathrm{Ch}(R)^{\mathrm{simp}}$ really has values in Kan complexes (this follows from it having values in simplicial groups, and it can be shown that any simplicial group is a Kan complex). Thus $\mathrm{Ch}(R)$ is Kan complex enriched and thus we define the ∞-category of chain complexes:

$$\mathcal{K}(R) \coloneqq \mathfrak{N}(\mathrm{Ch}(R))$$

There is ∞-categorical version of localization; it will turn out that localizing at quasi-isomorphisms yields

$$\mathcal{D}(R) \coloneqq \mathcal{K}(R)\big[\{\text{quasi-iso}\}^{-1}\big]$$

the derived ∞-category of the ring R.

- More general examples include e.g. *dg*-categories (see dg-categories).

For the remainder of this section: see the handwritten notes.

2 ∞-categorical abstract Nonsense

I can illustrate the ... approach with the ... image of a nut to be opened. The first analogy that came to my mind is of immersing the nut in some softening liquid, and why not simply water? From time to time you rub so the liquid penetrates better, and otherwise, you let time pass. The shell becomes more flexible through weeks and months — when the time is ripe, hand pressure is enough, the shell opens like a perfectly ripened avocado! A different image came to me a few weeks ago. The unknown thing to be known appeared to me as some stretch of earth or hard marble, resisting penetration ... the sea advances insensibly in silence, nothing seems to happen, nothing moves, the water is so far off you hardly hear it ... yet finally it surrounds the resistant substance.

Grothendieck

2.1 Universal Definitions and Examples

In the last few sessions we have devoted quite a lot of effort to defining ∞-categorical hom spaces, or more generally even the ∞-categorical hom-functor associated to an ∞-category $\mathscr{C}$. We were able to give such a construction due to the homotopy coherent nerve functor which fits into a Quillen equivalence with the rigidification functor:

$$\begin{tikzcd} (\mathrm{Set}_\Delta)_{\mathrm{Joyal}} \arrow[r, shift left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & (\mathrm{Cat}_\Delta)_{\mathrm{DKB}} \arrow[l, shift left, "\mathfrak{N}"] \end{tikzcd}$$

Hence it was that we defined the Yoneda embedding $\mathcal{Y}_{\mathscr{C}} \colon \mathscr{C} \to \mathcal{P}(\mathscr{C})$ by currying the $(\mathfrak{C} \dashv \mathfrak{N})$-adjunct of the composite

$$\begin{tikzcd}
\mathfrak{C}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C}) \arrow[r, "\text{DK equivalence}"] \arrow[d, dotted] & \mathfrak{C}(\mathscr{C})^{\mathrm{op}} \times \mathfrak{C}(\mathscr{C}) \arrow[d, "\mathfrak{C}(\mathscr{C})(-,-)"] \\
\mathrm{Kan} & \mathrm{sSet} \arrow[l, "\psi_\infty \circ |-|"]
\end{tikzcd}$$

Having established such a functor and the fact that there is a Yoneda Lemma also for ∞-categories immediately tells us that there ought to be (homotopy coherent) universal constructions just waiting to be defined. Before going there, let us quickly talk about (set-theoretical) size issues: Let κ be an infinite cardinal.

1. A simplicial set $X_\bullet$ is called κ-small, if the collection of non-degenerate simplices of X is κ-small.

2. An ∞-category is called essentially κ-small, if it is (Joyal) equivalent to a κ-small simplicial set.

3. An ∞-category $\mathscr{C}$ is called locally κ-small if, for every pair of objects $c, \tilde{c} \in \mathscr{C}$, the ∞-groupoid of morphisms $\mathscr{C}(c, \tilde{c})$ is essentially κ-small.

One can show that an ∞-category $\mathscr{C}$ is essentially κ-small if and only if it is locally κ-small and the set of isomorphism classes $\pi_0(\mathscr{C}^{\simeq})$ is κ-small. One is therefore reduced to the problem of testing essential κ-smallness of Kan complexes. However, it is shown that a Kan complex X is essentially κ-small if and only if the set $\pi_0(X)$ is κ-small and the homotopy groups $\{\pi_n(X, x)\}_{n>0}$ are κ-small for every vertex $x \in X$. For more on this see Kerodon. Having all that, let us move on to defining adjunctions of ∞-categories. Let $R \colon \mathscr{D} \to \mathscr{C}$ be a functor between ∞-categories.

1. Given objects $c \in \mathscr{C}, d \in \mathscr{D}$, and a morphism $\eta \colon c \to Rd$ in $\mathscr{C}$, we say η witnesses d as a left adjoint object to c under R if the composite

$$\mathscr{D}(d, -) \xrightarrow{\ R\ } \mathscr{C}(Rd, R(-)) \xrightarrow{\ \eta^\star\ } \mathscr{C}(c, R(-))$$

   is an equivalence of functors $\mathscr{D} \to \infty\mathrm{Grpd}$. Morally $d \simeq Lc$, for L a pointwise left adjoint.

2. An adjunction between R and a functor $L \colon \mathscr{C} \to \mathscr{D}$ is an equivalence

$$\mathscr{D}(L(-), -) \qquad \simeq \qquad \mathscr{C}(-, R(-))$$

An amazing consequence of the Yoneda Lemma is that in order to define left adjoint functors, it suffices to define them merely on objects (something that is of course really, really false for arbitrary functors). A functor $R\colon \mathscr{D} \to \mathscr{C}$ admits a left adjoint if and only if every $c \in \mathscr{C}$ admits a left adjoint object under R. More generally, if $\mathscr{C}_R \hookrightarrow \mathscr{C}$ is the full subcategory spanned by those objects $c \in \mathscr{C}$ which admit a left adjoint object, extracting these left-adjoint objects defines a functor

$$L\colon \mathscr{C}_L \to \mathscr{D}$$

Proof. By currying the functor

$$\mathscr{C}(-, R(-))\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \infty\mathrm{Grpd}$$

we obtain a functor $\tilde{R}\colon \mathscr{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{D}, \infty\mathrm{Grpd})$, which, when restricted to $\mathscr{C}_R$, lands in the representables, i.e., in the essential image of the Yoneda embedding $\mathcal{Y}_{\mathscr{D}}\colon \mathscr{D}^{\mathrm{op}} \to \mathrm{Fun}(\mathscr{D}, \infty\mathrm{Grpd})$. But $\mathcal{Y}_{\mathscr{D}}$ is fully faithful, hence an equivalence onto its essential image. Composing $\tilde{R}|_{\mathscr{C}_R^{\mathrm{op}}}$ with an inverse of this equivalence yields a functor $\mathscr{C}_R^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$ and hence a functor $L\colon \mathscr{C}_R \to \mathscr{D}$ after taking $(-)^{\mathrm{op}}$. But then by construction, $\mathcal{Y}_{\mathscr{D}} \circ L^{\mathrm{op}}$ is equivalent to $\tilde{R}|_{\mathscr{C}_R^{\mathrm{op}}}$ in $\mathrm{Fun}(\mathscr{C}_R^{\mathrm{op}}, \mathrm{Fun}(\mathscr{D}, \infty\mathrm{Grpd}))$. But this means $\mathscr{D}(L(-), -)$ and $\mathscr{C}(-, R(-))$ are equivalent in $\mathrm{Fun}(\mathscr{C}_R^{\mathrm{op}} \times \mathscr{D}, \infty\mathrm{Grpd})$. $\square$

We start off with some rather abstract, non-concrete examples before moving on to concrete computations: Let us start off with probably the most general kind of adjunction: the mysterious case of Kan extensions. For $\psi\colon \mathscr{E} \to \mathscr{C}$ a functor between ∞-categories, we recall that, for all ∞-categories $\mathscr{A}$, precomposition with ψ induces a functor

$$\psi^\star\colon \mathrm{Fun}(\mathscr{C}, \mathscr{A}) \to \mathrm{Fun}(\mathscr{E}, \mathscr{A})$$

If this functor has a left resp. right adjoint functor Lan_ψ resp. Ran_ψ, then these are called left Kan extension along ψ resp. right Kan extension along ψ. In the algebraic geometry community these are also typically denoted by $\psi_! \coloneqq \mathrm{Lan}_\psi$ and $\psi_\star \coloneqq \mathrm{Ran}_\psi$. Sticking to that notation, the whole triple adjunction reads:

$$\begin{tikzcd} \mathrm{Fun}(\mathscr{C}, \mathscr{A}) \arrow[r, "\psi^\star" description] & \mathrm{Fun}(\mathscr{E}, \mathscr{A}) \arrow[l, bend right, "\psi_!"'] \arrow[l, bend left, "\psi_\star"] \end{tikzcd}$$

Let $\mathscr{I}, \mathscr{C}$ be ∞-categories and consider the unique morphism $!\colon \mathscr{I} \to \Delta^0$ (note that Δ^0 is the terminal ∞-category). Precomposition with ! induces the constant diagram functor

$$\mathrm{const}\colon \mathscr{C} \to \mathrm{Fun}(\mathscr{I}, \mathscr{C})$$

The limit and colimit functors, if they exist, are then defined to be right resp. left adjoints to const:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "\mathrm{const}" description] & \mathrm{Fun}(\mathscr{I}, \mathscr{C}) \arrow[l, bend right, "\operatorname{colim}_{\mathscr{I}}"'] \arrow[l, bend left, "\lim_{\mathscr{I}}"] \end{tikzcd}$$

In fact, as we already know from ordinary category theory, this is just a special case of the above example by putting ψ to be the unique morphism $\mathscr{I} \to \Delta^0$. Let $f\colon \mathscr{I} \to \infty\mathrm{Grpd}$ be a functor. Let us try to compute its limit by using some formal abstract nonsense. If we assumed for a moment that such a limit in ∞Grpd existed, then we could calculate

$$\begin{aligned}\lim_{\mathscr{I}} f &\simeq \infty\mathrm{Grpd}(\Delta^0, \lim_{\mathscr{I}} f) \\ &\simeq \mathrm{Fun}(\mathscr{I}, \infty\mathrm{Grpd})(\mathrm{const}(\Delta^0), f)\end{aligned}$$

Putting $\lim\limits_{\mathscr{I}} f := \mathrm{Fun}(\mathscr{I}, \infty\mathrm{Grpd})(\mathrm{const}(\Delta^0), f)$ yields a pointwise right adjoint object in the sense of 2.1. There is two ways (I am aware of) one could go about proving this: either we use that ∞Grpd is the free ∞-cocompletion of the point Δ^0, or we decompose mapping spaces by means of ∞-ends - we will not spell out either of these, so you have to take my word on this. Hence $\lim\limits_{\mathscr{I}}$ extends to a functor $\mathrm{Fun}(\mathscr{I}, \mathscr{C}) \to \mathscr{C}$ by Corollary 2.1. Let us illustrate the previous example in the space case where $\mathscr{I} = \Lambda^2_2$ (the pullback shape). A functor $f\colon \Lambda^2_2 \to \infty\mathrm{Grpd}$ is the same as a diagram of 1-morphisms in ∞Grpd of the form:

$$\begin{tikzcd} & b \arrow[d, "f_b"] \\ a \arrow[r, "f_a"'] & c \end{tikzcd}$$

The objects of $a \times_c b := \lim\limits_{\Lambda^2_2} f$ are maps $\Delta^1 \times \Lambda^2_2 \to \infty\mathrm{Grpd}$ which, under restriction, will be $\mathrm{const}(\Delta^0)$ resp. f. A general map of shape $\Delta^1 \times \Lambda^2_2$ has its image looking like

$$\begin{tikzcd} B \arrow[rrr] \arrow[drr, phantom, "\Swarrow"] \arrow[ddr] & & & \tilde{B} \arrow[ddl] \\ & A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \tilde{A} \arrow[d] & \\ & C \arrow[r] & \tilde{C} & \end{tikzcd}$$

and adjusting this to our case, an object $\mathrm{const}(\Delta^0) \to f$ in $a \times_c b$ is the same as a diagram

$$\begin{tikzcd} \Delta^0 \arrow[rrr, "\mathrm{pt}_b"] \arrow[drr, phantom, "\Swarrow"] \arrow[ddr] & & & b \arrow[ddl, "f_b"] \\ & \Delta^0 \arrow[r, "\mathrm{pt}_a"] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & a \arrow[d, "f_a"] & \\ & \Delta^0 \arrow[r, "\mathrm{pt}_c"'] & c & \end{tikzcd}$$

But the above diagram just says that we have paths connecting the points $f_b(\mathrm{pt}_b) \simeq \mathrm{pt}_c \simeq f_a(\mathrm{pt}_a)$. Hence, in particular we have

$$(a \times_c b)_0 = \{(\Delta^0 \overset{\mathrm{pt}_a}{\to} a,\ \Delta^0 \overset{\mathrm{pt}_b}{\to} b,\ \Delta^1 \overset{\gamma}{\to} c)\colon [s(\gamma) = f_a(\mathrm{pt}_a)] \wedge [t(\gamma) = f_b(\mathrm{pt}_b)]\}$$

Moreover, even though we have specified the pullback functor

$$\lim_{\Lambda^2_2}\colon \mathrm{Fun}(\Lambda^2_2, \infty\mathrm{Grpd}) \to \infty\mathrm{Grpd}$$

merely on objects, we know by Corollary 2.1 that this really extends to a functor. It is a good exercise to try to compute the equaliser for two morphisms of ∞-groupoids. A more general example of an adjunction is the following: Consider the fully faithful embedding

$$\infty\mathrm{Grpd} \hookrightarrow \infty\mathrm{Cat}$$

induced by the canonical embedding of simplicially enriched categories $\mathrm{Kan} \hookrightarrow \mathrm{QCat}^{\simeq}$ after applying the homotopy coherent nerve $\mathfrak{N}$. This functor has both a left adjoint $|-|\colon \infty\mathrm{Cat} \to \infty\mathrm{Grpd}$, which sends an ∞-category to its localization at all the morphisms, and a right adjoint $\mathrm{core}\colon \infty\mathrm{Cat} \to \infty\mathrm{Grpd}$ which throws away all non-invertible 1-morphisms. In total,

$$\begin{tikzcd} \infty\mathrm{Grpd} \arrow[r, hook, "\perp", "\perp"'] & \infty\mathrm{Cat} \arrow[l, shift right=6, "|-|"'] \arrow[l, shift left=6, "\mathrm{core}"] \end{tikzcd}$$

If we use relative categories as a model for ∞-categories, then the adjunction above is induced by the homotopical adjunction

$$\begin{tikzcd} (\mathrm{Kan})_{\mathrm{Quillen}} \arrow[r, hook, "\perp", "\perp"'] & (\mathrm{QCat})_{\mathrm{Joyal}} \arrow[l, shift right=6] \arrow[l, shift left=6] \end{tikzcd}$$

Taking the left and right derived functors of the left resp. right adjoint leads to the correct functors (uniquely determined up to equivalence). We spell out this relationship more carefully in the next section. Consider the ordinary category $\mathrm{Ch}_\bullet(R\mathrm{mod})$ for some ring R. The derived ∞-category of the ring R is defined to be the pushout

$$\begin{tikzcd} \mathscr{W}_{\{\text{quasi-isos}\}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{po}"] & \mathrm{Ch}_\bullet(R\mathrm{mod}) \arrow[d, dotted] \\ |\mathscr{W}_{\{\text{quasi-isos}\}}| \arrow[r, dotted] & \mathscr{D}(R) \end{tikzcd}$$

2.2 Homotopy Kan extensions

[label=Author]Lurie You might be tempted to think this is theory for theory's sake. It's not. It's theory for the sake of other theory...

2.2.1 A reminder

We recall the basic idea of a derived functor in a homotopical category $(\mathscr{C}, \mathscr{W}_{\mathscr{C}})$. Later on we will relate this notion with the notion of an adjunction between ∞-categories. Let $f\colon (\mathscr{C}, \mathscr{W}_{\mathscr{C}}) \to (\mathscr{D}, \mathscr{W}_{\mathscr{D}})$ be a functor between homotopical categories. A homotopical functor $\mathbb{L}f\colon (\mathscr{C}, \mathscr{W}_{\mathscr{C}}) \to (\mathscr{D}, \mathscr{W}_{\mathscr{D}})$ along with a comparison natural transformation $l\colon \mathbb{L}f \to f$ is called a left derived functor if it induces a right Kan extensions as follows:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "f"] \arrow[d, two heads, "\gamma"'] & \mathscr{D} \arrow[r, two heads, "\delta"] \arrow[dl, Rightarrow, "\delta l"] & \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}] \\
\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \arrow[urr, "\mathrm{loc}(\delta\circ\mathbb{L}f)"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{C} \arrow[r, "\delta f"] \arrow[d, two heads, "\gamma"'] \arrow[d, phantom, shift left=4ex, "\Downarrow"] & \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}] \\
\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \arrow[ur, dotted, "\mathrm{Ran}_{\gamma}(\delta f)"'] &
\end{tikzcd}$$

where $\mathrm{loc}(\delta \circ \mathbb{L}f)$ is the unique functor coming from the universal property of the localization:

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\mathbb{L}f"] \arrow[d, two heads, "\gamma"'] & \mathscr{D} \arrow[d, two heads, "\delta"] \\
\mathscr{C}[\mathscr{W}_{\mathscr{C}}^{-1}] \arrow[r, dotted, "\mathrm{loc}(\delta\circ\mathbb{L}f)"'] & \mathscr{D}[\mathscr{W}_{\mathscr{D}}^{-1}]
\end{tikzcd}$$

One can dualize this of course to arrive at the notion of right derived functors. In particular, $\mathbf{L}f \coloneqq \mathrm{Ran}_{\gamma}(\delta f)$ is called the total left derived functor of f. In the previous sessions we then had existence results for such left/right derived functors. Namely, a sufficient condition for the to-be derived functor should be left/right deformability. In the presence of a model structure, a particularly good pair of candidates to derive is a Quillen adjunction. Let us try to derive the procedure of taking Kan extensions along some functor $\psi\colon \mathscr{I} \to \mathscr{J}$. Let $\mathscr{M}$ be a model category. Then the functor category $\mathscr{M}^{\mathscr{I}}$ has a homotopical structure where a natural transformation is a weak equivalence if and only if each of its components is a weak equivalence. In fact, in good cases the functor category might even give rise to two dual model structures:

- The projective model structure, where fibrations in $\mathscr{M}^{\mathscr{I}}_{\mathrm{proj}}$ are natural transformations which are componentwise fibrations.
- The injective model structure, where cofibrations in $\mathscr{M}^{\mathscr{I}}_{\mathrm{inj}}$ are natural transformations which are componentwise cofibrations.

Recall that $\mathrm{Lan}_\psi \dashv \psi^\star \dashv \mathrm{Ran}_\psi$, and furthermore note that $\psi^\star$ is both a left resp. right Quillen functor as follows (it clearly preserves componentwise (trivial) cofibrations resp. (trivial) fibrations:

$$\begin{tikzcd} \mathscr{M}^{\mathscr{I}}_{\mathrm{inj}} \arrow[r, shift left=2, "\psi^\star"] \arrow[r, phantom, "\perp\text{Quillen}"] & \mathscr{M}^{\mathscr{I}}_{\mathrm{inj}} \arrow[l, shift left=2, "\mathrm{Ran}_\psi"] \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{M}^{\mathscr{I}}_{\mathrm{proj}} \arrow[r, shift right=2, "\psi^\star"'] \arrow[r, phantom, "\perp\text{Quillen}"] & \mathscr{M}^{\mathscr{I}}_{\mathrm{proj}} \arrow[l, shift right=2, "\mathrm{Lan}_\psi"'] \end{tikzcd}$$

Hence these pairs are derivable, leading to homotopy Kan extension functors along ψ:

$$\mathrm{hoRan}_\psi \colon \mathscr{M}^{\mathscr{I}} \to \mathscr{M}^{\mathscr{I}}, \qquad \mathrm{hoLan}_\psi \colon \mathscr{M}^{\mathscr{I}} \to \mathscr{M}^{\mathscr{I}}$$

In particular, in the case of limits this yields homotopy (co)limit functors:

$$\operatorname*{holim}_{\mathscr{I}} \colon \mathscr{M}^{\mathscr{I}} \to \mathscr{M}, \qquad \operatorname*{hocolim}_{\mathscr{I}} \colon \mathscr{M}^{\mathscr{I}} \to \mathscr{M}$$

2.2.2 Warm-up Intuition

A homtopy pullback should really model the same thing as the ∞-pullback from example 2.1. That is, for a category $\mathscr{M}$ with a notion of homotopy (e.g. model categories) the homotopy pullback of a diagram $a \to c \leftarrow b$ in $\mathscr{M}$ assembles into a diagram

$$\begin{tikzcd} a \times^h_c b \arrow[r, dotted] \arrow[d, dotted] & b \arrow[d] \\ a \arrow[r] & c \end{tikzcd}$$

in $\mathscr{M}$ which commutes only up to homotopy. Moreover, the homotopy pullback is homotopically terminal in the following manner:

$$\begin{tikzcd} P \arrow[dr, dashed, "\exists"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & a \times^h_c b \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\ & a \arrow[r] & c \end{tikzcd}$$

In other words, we have an equivalence

$$\mathrm{Map}(P, a \times^h_c b) \simeq \mathrm{HoSq}(P, a \to c \leftarrow b)$$

between maps from P into the homotopy pullback with the space of homtopy commutative squares with vertex P. The above is sort of formalized by the derived adjunction:

$$\begin{tikzcd} \mathrm{Ho}(\mathscr{M}^{\Lambda_2^2}) \arrow[r, shift right, "\mathbf{R}\lim_{\Lambda_2^2}"'] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\mathscr{M} \arrow[l, shift right, "\mathbf{L}\mathrm{const}"'] \end{tikzcd}$$

or more generally the adjunction

$$\begin{tikzcd} \mathrm{Ho}(\mathscr{M}^{\mathscr{I}}) \arrow[r, shift right, "\mathbf{R}\lim_{\mathscr{I}}"'] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\mathscr{M} \arrow[l, shift right, "\mathbf{L}\mathrm{const}"'] \end{tikzcd}$$

2.2.3 When is an ordinary pullback a homotopy pullback?

We recall that in order to calculate e.g. the right derived functor of $\lim_{\mathscr{I}}$ all we have to do is to precompose with a right deformation

$$R\colon \mathscr{M}^{\mathscr{I}} \to \mathscr{M}^{\mathscr{I}}$$

on whose image $\lim_{\mathscr{I}}$ is homotopical. In the case where we apply the right derived functor to an already fibrant object, the image will be equivalent to the image of that object under the non-derived functor (this follows from Ken Brown's Lemma). So there are instances where e.g. the ordinary limit coincides with the homotopy limit. Let us try to figure out, in the specific situation of pullbacks, when a homotopy pullback may be written as a strict pullback. We have the following result: If $a \xrightarrow{f} c \xleftarrow{g} b$ is such that all objects are fibrant, then the ordinary pullback equals the homotopy pullback if one of the two morphisms is a fibration. Using this Lemma we can prove the following: If $a \xrightarrow{f} c \xleftarrow{g} b$ is such that all objects are fibrant, then the homotopy pullback is given by the ordinary pullback

$$\begin{tikzcd} a \times_c^h b \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{Path}(c) \arrow[d, two heads] \\ a \times b \arrow[r, "f \times g"'] & c \times c \end{tikzcd}$$

where $\mathrm{Path}(c) \to c \times c$ is a fibrant resolution of the diagonal map $c \to c \times c$.

Proof. First of all we note that any pullback (on the left) may be rewritten as depicted (on the right):

$$\begin{tikzcd} \bullet \arrow[r, dotted] \arrow[d, dotted] & x \arrow[d, "g"'] \\ y \arrow[r, "f"'] & z \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, dotted] \arrow[d, dotted] & c \arrow[d, "\Delta"] \\ x \times y \arrow[r, "f \times g"'] & c \times c \end{tikzcd}$$

where $\Delta\colon c \to c \times c$ is the diagonal map obtained from the universal property of the product $c \times c$. Hence in order to compute the homotopy pullback, we really only have to fibrantly replace one of the two maps (by the above Lemma). We choose to replace the diagonal map and factorize it (by axioms of our model category structure) as

$$\begin{tikzcd} c \arrow[r, tail, "\sim"] \arrow[d, "\Delta"'] & \mathrm{Path}(c) \arrow[dl, two heads, "(p_1,p_0)"] \\ c \times c & \end{tikzcd}$$

where $\mathrm{Path}(c)$ is referred to as a path space object (and is usually obtained by taking the internal hom with an interval object). All in all, this leads to homotopy equivalent diagrams and since the homotopy limit preserves equivalences of that sort, we can simply calculate the homotopy limit of the diagram

$$\begin{tikzcd} & \mathrm{Path}(c) \arrow[d, two heads, "(p_1,p_0)"] \\ a \times b \arrow[r, "f \times b"'] & c \times c \end{tikzcd}$$

But since all the objects are fibrant and one of the two morphisms is a fibration, the Lemma implies this computes the homotopy limit. $\square$

Let us calculate something in the Quillen model structure of simplicial sets. Fix a diagram

$$\begin{tikzcd} & c \arrow[d, "\Delta"] \\ a \times b \arrow[r, "f \times g"'] & c \times c \end{tikzcd}$$

consisting of Kan complexes. Let us calculate a fibrant resolution of the diagonal. We have an obvious factorization:

$$\begin{tikzcd} c \arrow[r, tail, shift left, "\sim"] \arrow[d, "\Delta"'] & \mathrm{Fun}(\Delta^1, c) \arrow[l, shift left, "r"] \arrow[dl, two heads, "(d_1^\star,d_0^\star)"] \\ c \times c & \end{tikzcd}$$

where the first map is $\mathrm{Fun}(\Delta^0, c) \xrightarrow{s_0^\star} \mathrm{Fun}(\Delta^1, c)$ is a trivial cofibration, while $(d_1^\star, d_0^\star)$ is shown to be a fibration. By the above proposition, we therefore have

$$\begin{tikzcd} a \times_c^h b \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{Fun}(\Delta^1, c) \arrow[d, two heads, "(d_1^\star,d_0^\star)"] \\ a \times b \arrow[r, "f \times g"'] & c \times c \end{tikzcd}$$

This precisely recovers Example 2.1. We can do the same thing for pushouts, but here we use the other kind of factorization of the codiagonal map $\nabla\colon c\coprod c \to c$:

$$\begin{tikzcd} c\coprod c \arrow[r, tail] \arrow[d, "\nabla"'] & \mathrm{Cyl}(c) \arrow[dl, two heads, "\sim"] \\ c & \end{tikzcd}$$

Then in order to compute the homotopy pushout of

$$\begin{tikzcd} c\coprod c \arrow[r, "f\coprod g"] \arrow[d, "\nabla"'] & a\coprod b \\ c & \end{tikzcd}$$

we replace the codiagonal as in the factorization by $c\coprod c \to \mathrm{Cyl}(c)$ (here $\mathrm{Cyl}(c)$ is referred to as the cylinder object) and then take the ordinary pushout. Specifically for simplicial sets (Quillen model structure) this will be a pushout like:

$$\begin{tikzcd} c\coprod c \arrow[r, "f\coprod g"] \arrow[d, tail, "(i_0,i_1)"'] \arrow[dr, phantom, "\text{po}"] & a\coprod b \arrow[d] \\ c\times\Delta^1 \arrow[r] & a\coprod_c^h b \end{tikzcd}$$

where the cylinder object $\mathrm{Cyl}(c) = c\times\Delta^1$.

2.2.4 Homotopy Kan extensions in Kan complex enriched categories

Since we know how to compute homotopy Kan extensions in spaces or Kan complexes, it is not too far fetched to believe there is a way to also extend this definition to general Kan complex enriched categories.

Recall that for $f\colon \mathscr{C}\to\mathscr{D}$ a functor between ordinary categories, its ordinary limit $\lim\limits_{\mathscr{C}} f$ is characterized by the fact that for every object $d\in\mathscr{D}$ we have

$$\mathscr{D}(d, \lim_{\mathscr{C}} f) \simeq \lim_{\mathscr{C}} \mathscr{D}(d, fc)$$

The analogous statement here is that all homotopy (co)limits are determined by homotopy limits in $(\mathrm{Set}_\Delta)_{\mathrm{Quillen}}$. In particular, since $(\mathrm{Set}_\Delta)_{\mathrm{Quillen}}$ models ∞-groupoids, this means that general ∞-limits are determined by ∞-limits in spaces or ∞-groupoids. Recall that, intrinsically, Kan extensions, as every universal construction, are supposed to be only defined up to weak equivalence, it is useful to make the extra freedom of choosing any weakly equivalent object explicit by the following definition: For $\varphi \in \mathscr{M}_{\mathrm{inj}}^{\mathscr{I}}$ and $\varphi' \in \mathscr{M}_{\mathrm{inj}}^{\mathscr{J}}$ and a morphism $\vartheta \colon \varphi' \to \mathrm{Ran}_{\psi}(\varphi)$, we say that ϑ exhibits φ' as a homotopy right Kan extension of φ if for some injectively fibrant resolution $\varphi \xrightarrow{\sim} R(\varphi)$ the composite morphism

$$\begin{tikzcd}
\varphi' \arrow[r, "\vartheta"] \arrow[dr, dotted, "\sim"'] & \mathrm{Ran}_{\psi}(\varphi) \arrow[d] \\
& \mathrm{Ran}_{\psi}(R(\varphi))
\end{tikzcd}$$

This is certainly dualizable, that is, we can dualize the definition so as to get something analogous for homotopy left Kan extensions (in this case we consider the projective model category structure). We can then make the following definition for Kan complex enriched categories. For $\mathscr{F}$ a Kan complex enriched category and $\psi \colon \mathscr{C} \to \mathscr{C}'$ an enriched functor of small simplicially enriched categories, given $\varphi \in \mathscr{F}^{\mathscr{C}}$ and $\varphi' \in \mathscr{F}^{\mathscr{C}'}$, we say a morphism $\vartheta \colon \varphi' \to \mathrm{Ran}_{\psi}(\varphi)$ exhibits φ' as a homotopy right Kan extension if for all $x \in \mathscr{F}$ the morphism

$$\vartheta_{\star} \colon \mathscr{F}(x, \varphi'(-)) \to \mathscr{F}(x, \mathrm{Ran}_{\psi}(\varphi)(-))$$

exhibits $\mathscr{F}(x, \varphi'(-)) \colon \mathscr{C}' \to (\mathrm{sSet})_{\mathrm{Quillen}}$ as a homotopy right Kan extension of $\mathscr{F}(x, \varphi(-)) \colon \mathscr{C} \to (\mathrm{Set}_{\Delta})_{\mathrm{Quillen}}$ along ψ:

$$\begin{tikzcd}
\mathscr{F}(x, \varphi'(-)) \arrow[r, "\vartheta_{\star}"] \arrow[d, dotted, "\sim"'] & \mathscr{F}(x, \mathrm{Ran}_{\psi}(\varphi)(-)) \arrow[d, "\text{induced by UP of target}"] \\
\mathrm{hoRan}_{\psi}\Big(\mathscr{F}(x, \varphi(-))\Big) & \mathrm{Ran}_{\psi}\Big(\mathscr{F}(x, \varphi(-))\Big) \arrow[l, "\text{fibrant resolution}"']
\end{tikzcd}$$

For $\mathscr{F}$ a Kan complex enriched category and $\psi \colon \mathscr{C} \to \mathscr{C}'$ an enriched functor of small simplicially enriched categories, given $\varphi \in \mathscr{F}^{\mathscr{C}}$ and $\varphi' \in \mathscr{F}^{\mathscr{C}'}$, then a morphism $\vartheta \colon \varphi' \to \mathrm{Ran}_{\psi}(\varphi)$ exhibits φ' as a homotopy right Kan extension if and only if the homotopy coherent nerve takes φ' to the ∞-right Kan extension $\mathrm{Ran}_{\mathfrak{N}\psi}(\mathfrak{N}\varphi)$. The ∞-categories of ∞-groupoids and ∞-categories are both complete and cocomplete.